GENERALIZATION OF THE GRIFFITH-SNEDDON CRITERION FOR THE CASE OF A NONHOMOGENEOUS BODY

(OBOBSHCHENIE KRITERIIA GRIFFITSA-SNEDDONA NA SLUCHAI NEODNORODNOGO TELA)

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This paper presents the solution to the problem in theory of elasticity for two connected half-spaces with a circular crack in the plane of joining. Starting with the Griffith criterion, the value for critical stress is found.

1. Formulation of problem and derivation of boundary conditions. We consider the problem of elastic half-spaces with different elastic properties. In the plane connecting these half-spaces there is a circular crack of radius a. At infinity a tensile stress p = const is applied perpendicular to the plane of the crack. Rectangular coordinates are chosen so that the boundary of the elastic half-spaces coincides with the plane $z = 0$; the origin is located in the center of the crack (Fig.1).

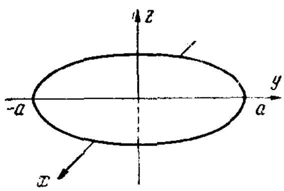

Fig. 1

The solution of this problem is sought in the form [1]

$$u = \varphi_1 + z\frac{\partial\psi}{\partial x}, \qquad v = \varphi_2 + z\frac{\partial\psi}{\partial y}$$

$$w = \varphi_3 + z\frac{\partial\psi}{\partial z} \tag{1.1}$$

Here $u(x, y, z)$, $v(x\ y, z)$, $w(x, y, z)$ are projections of elastic displacements on the axes of the rectangular coordinates; φ_1, φ_2, φ_3 and ψ are space functions of x, y, z, connected by relation

$$\frac{\partial\psi}{\partial z} = \frac{1}{4\nu - 3}\left(\frac{\partial\varphi_1}{\partial x} + \frac{\partial\varphi_2}{\partial y} + \frac{\partial\varphi_3}{\partial z}\right) \tag{1.2}$$

where ν is Poisson's ratio.

Expressing the stresses in terms of deformations, using Equations (1.1), we find the components of the stress tensor σ_z, τ_{xz}, τ_{yz} on the plane $z = 0$

$$\sigma_z\,(x,\,y,\,0) = (\lambda + 2\mu)\frac{\partial}{\partial z}(\varphi_3 + \psi) + \lambda\left(\frac{\partial\varphi_1}{\partial x} + \frac{\partial\varphi_2}{\partial y}\right) \qquad (1.3)$$

$$\tau_{xz}\,(x,\,y,\,0) = \mu\left(\frac{\partial\varphi_1}{\partial z} + \frac{\partial\varphi_3}{\partial x} + \frac{\partial\psi}{\partial x}\right), \qquad \tau_{yz}\,(x,\,y,\,0) = \mu\left(\frac{\partial\varphi_2}{\partial z} + \frac{\partial\varphi_3}{\partial y} + \frac{\partial\psi}{\partial y}\right)$$

where λ and μ are the Lamé coefficients. From (1.1), placing $z = 0$, we obtain

$$u\,(x,\,y,\,0) = \varphi_1\,(x,\,y,\,0), \qquad v\,(x,\,y,\,0) = \varphi_2\,(x,\,y,\,0)$$
$$w\,(x,\,y,\,0) = \varphi_3\,(x,\,y,\,0) \qquad (1.4)$$

All quantities related to the upper half-space are designated by plus, to the lower by minus.

Boundary conditions for the problem are as follows. On the crack from above and below are normal stresses σ_z^+ and σ_z^- of value p, tangential stresses are absent. Exterior to the crack the half-spaces are welded. Taking into account Equations (1.3) and (1.4), the condition for the determination of unknown functions φ_1, φ_2, φ_3 and ψ may be written in the form

$$(\lambda_1 + 2\mu_1)\frac{\partial}{\partial z}(\varphi_3^+ + \psi^+) + \lambda_1\left(\frac{\partial\varphi_1^+}{\partial x} + \frac{\partial\varphi_2^+}{\partial y}\right) = p, \qquad \mu_1\left(\frac{\partial\varphi_1^+}{\partial z} + \frac{\partial\varphi_3^+}{\partial x} + \frac{\partial\psi^+}{\partial x}\right) = 0$$

$$(\lambda_2 + 2\mu_2)\frac{\partial}{\partial z}(\varphi_3^- + \psi^-) + \lambda_2\left(\frac{\partial\varphi_1^-}{\partial x} + \frac{\partial\varphi_2^-}{\partial y}\right) = p, \qquad \mu_2\left(\frac{\partial\varphi_1^-}{\partial z} + \frac{\partial\varphi_3^-}{\partial x} + \frac{\partial\psi^-}{\partial x}\right) = 0$$

$$\mu_1\left(\frac{\partial\varphi_2^+}{\partial z} + \frac{\partial\varphi_3^+}{\partial y} + \frac{\partial\psi^+}{\partial y}\right) = 0, \qquad \mu_2\left(\frac{\partial\varphi_2^-}{\partial z} + \frac{\partial\varphi_3^-}{\partial y} + \frac{\partial\psi^-}{\partial y}\right) = 0$$

$$(\text{Interior to crack } \rho = \sqrt{x^2 + y^2} < a) \qquad (1.5)$$

$$(\lambda_1 + 2\mu_1)\frac{\partial}{\partial z}(\varphi_3^+ + \psi^+) + \lambda_1\left(\frac{\partial\varphi_1^+}{\partial x} + \frac{\partial\varphi_2^+}{\partial y}\right) =$$
$$= (\lambda_2 + 2\mu_2)\frac{\partial}{\partial z}(\varphi_3^- + \psi^-) + \lambda_2\left(\frac{\partial\varphi_1^-}{\partial x} + \frac{\partial\varphi_2^-}{\partial y}\right)$$
$$\mu_1\left(\frac{\partial\varphi_1^+}{\partial z} + \frac{\partial\varphi_3^+}{\partial x} + \frac{\partial\psi^+}{\partial x}\right) = \mu_2\left(\frac{\partial\varphi_1^-}{\partial z} + \frac{\partial\varphi_3^-}{\partial x} + \frac{\partial\psi^-}{\partial x}\right)$$
$$\mu_2\left(\frac{\partial\varphi_2^+}{\partial z} + \frac{\partial\varphi_3^+}{\partial y} + \frac{\partial\psi^+}{\partial y}\right) = \mu_2\left(\frac{\partial\varphi_2^-}{\partial z} + \frac{\partial\varphi_3^-}{\partial y} + \frac{\partial\psi}{\partial y}\right)$$
$$\varphi_1^+ = \varphi_1^-, \qquad \varphi_2^+ = \varphi_2^-, \qquad \varphi_3^+ = \varphi_3^- \quad (\text{Exterior to crack } \rho = \sqrt{x^2 + y^2} > a) \qquad (1.6)$$

Here λ_1, μ_1 and λ_2, μ_2 are Lamé coefficients for the upper and lower half-spaces respectively. Let us introduce harmonic functions corresponding to the upper and lower half-spaces

$$\psi^+ = \frac{1}{4\nu_1 - 3}(\varphi_4^+ + \varphi_3^+), \qquad \psi^- = \frac{1}{4\nu_2 - 3}(\varphi_4^- + \varphi_3^-)$$
$$\left(\nu_k = \frac{\lambda_k}{2(\lambda_k + \mu_k)} \quad (k = 1,\,2)\right) \qquad (1.7)$$

From (1.2) and (1.7) it follows that

$$\frac{\partial\varphi_4^+}{\partial z} = \frac{\partial\varphi_1^+}{\partial x} + \frac{\partial\varphi_2^+}{\partial y}, \qquad \frac{\partial\varphi_4^-}{\partial z} = \frac{\partial\varphi_1^-}{\partial x} + \frac{\partial\varphi_2^-}{\partial y} \tag{1.8}$$

Using (1.2) and (1.8) , from the first and second of Equations (1.5) we obtain

$$\frac{\partial\varphi_3^+}{\partial z} - \frac{1}{A_1}\frac{\partial\varphi_4^+}{\partial z} = C_1, \qquad \frac{\partial\varphi_3^-}{\partial z} - \frac{1}{A_2}\frac{\partial\varphi_4^-}{\partial z} = C_2 \tag{1.9}$$

$$\left(A_k = \frac{\lambda_k + 2\mu_k}{\mu_k},\ C_k = \frac{\lambda_k + 3\mu_k}{2\mu_k(\lambda_k + 2\mu_k)}\,p \quad (k = 1,\ 2)\right)$$

Let us differentiate the third and the fifth of Equations (1.5) with respect to x and to y and add

$$\mu_1\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\varphi_3^+ + \psi^+) + \frac{\partial}{\partial z}\left(\frac{\partial\varphi_1^+}{\partial x} + \frac{\partial\varphi_2^+}{\partial y}\right)\right] = 0 \tag{1.10}$$

Thus functions φ_3 and ψ are harmonic, then

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\varphi_3^+ + \psi^+) = -\frac{\partial^2}{\partial z^2}(\varphi_3^+ + \psi^+) \tag{1.11}$$

Substituting (1.11) into (1.10) and taking into account (1.7) and (1.8) we obtain

$$\frac{\partial^2\varphi_3^+}{\partial z^2} + A_1\frac{\partial^2\varphi_4^+}{\partial z^2} = 0 \qquad (\rho < a) \tag{1.12}$$

Analogously, from the fourth and sixth of equations (1.5) we have

$$\frac{\partial^2\varphi_3^-}{\partial z^2} + A_2\frac{\partial^2\varphi_4^-}{\partial z^2} = 0 \qquad (\rho < a) \tag{1.13}$$

Using (1.2) and (1.8), we put the first equation of (1.6) in the form

$$B_1\frac{\partial\varphi_3^+}{\partial z} - D_1\frac{\partial\varphi_4^+}{\partial z} = B_2\frac{\partial\varphi_3^-}{\partial z} - D_2\frac{\partial\varphi_4^-}{\partial z} \qquad (\rho > a) \tag{1.14}$$

$$B_k = \frac{\mu_k(\lambda_k + 2\mu_k)}{\lambda_k + 3\mu_k}, \qquad D_k = \frac{\mu_k^2}{\lambda_k + 3\mu_k} \qquad (k = 1,\ 2) \tag{1.15}$$

Differentiating the second and third of Equations (1.6) by x and y respectively, adding and taking into account (1.7), we get

$$D_1\frac{\partial^2\varphi_3^+}{\partial z^2} - B_1\frac{\partial^2\varphi_4^+}{\partial z^2} = D_2\frac{\partial^2\varphi_3^-}{\partial z^2} - B_2\frac{\partial^2\varphi_4^-}{\partial z^2} \qquad (\rho > a) \tag{1.16}$$

Multiplying the first equation of (1.9) by B_1, the second by B_2, subtracting and taking into account that $B_1 / A_1 = D_1$, $B_2 / A_2 = D_2$, and $C_1B_1 - C_2B_2 = 0$, we get

$$B_1\frac{\partial\varphi_3^+}{\partial z} - D_1\frac{\partial\varphi_4^+}{\partial z} - B_2\frac{\partial\varphi_3^-}{\partial z} + D_2\frac{\partial\varphi_4^-}{\partial z} = 0 \qquad (\rho < a) \tag{1.17}$$

Analogously from (1.3) and (1,13) we find

$$D_1\frac{\partial^2\varphi_3^+}{\partial z^2} - B_1\frac{\partial^2\varphi_4^+}{\partial z^2} - D_2\frac{\partial^2\varphi_3^-}{\partial z^2} + B_2\frac{\partial^2\varphi_4^-}{\partial z^2} = 0 \qquad (\rho < a) \tag{1.18}$$

Integrating Equation (1.13) we obtain $\varphi_3^- - A_2\varphi_4^- = K_2(x, y)$, where $K_2(x, y)$ is an unknown harmonic function.

In the case under consideration the state of stress is axially symmetric, therefore functions φ_3^- and φ_4^- must be axially symmetric. Therefore,

$K_2(x,y) = C$ is a constant subject to determination.

Thus, boundaru conditions for the determination of φ_3 and φ_4 are

$$\frac{\partial\varphi_3^-}{\partial z} - \frac{1}{A_2}\frac{\partial\varphi_4^-}{\partial z} = C_2, \qquad \varphi_3^- - A_2\varphi_4^- = C \qquad (\rho < a) \qquad (1.19)$$

$$\varphi_3^+ = \varphi_3^-, \qquad \frac{\partial\varphi_4^+}{\partial z} = \frac{\partial\varphi_4^-}{\partial z} \qquad (\rho > a) \qquad (1.20)$$

$$\begin{aligned} D_1\frac{\partial^2\varphi_3^+}{\partial z^2} - D_2\frac{\partial^2\varphi_3^-}{\partial z^2} - B_1\frac{\partial^2\varphi_4^+}{\partial z^2} + B_2\frac{\partial^2\varphi_4^-}{\partial z^2} = 0 \\ B_1\frac{\partial\varphi_3^+}{\partial z} - B_2\frac{\partial\varphi_3^-}{\partial z} - D_1\frac{\partial\varphi_4^+}{\partial z} + D_2\frac{\partial\varphi_4^-}{\partial z} = 0 \end{aligned} \qquad (z = 0) \qquad (1.21)$$

We introduce the following functions for consideration:

$$\varphi_3^*(x, y, z) = \varphi_3(x, y, -z), \qquad \varphi_4^*(x, y, z) = \varphi_4(x, y, -z)$$

From relation (1.21) we have

$$\begin{aligned} \varphi_3^* = E\varphi_3^- - H\varphi_4^- \\ \varphi_4^* = H\varphi_3^- - E\varphi_4^- \end{aligned} \qquad \left(E = \frac{B_1B_2 + D_1D_2}{D_1^2 - B_1^2},\ H = \frac{B_1D_2 + B_2D_1}{D_1^2 - B_1^2}\right) \qquad (1.22)$$

Now the boundary conditions exterior to the crack are transformed to the form

$$\varphi_3^- - A_0\varphi_4^- = 0, \quad \frac{\partial\varphi_3^-}{\partial z} - \frac{1}{A_0}\frac{\partial\varphi_4^-}{\partial z} = 0 \qquad (\rho > a) \qquad \left(A_0 = \frac{H}{E-1}\right) \quad (1.23)$$

We introduce functions $F_1(x,y,z)$ and $F_2(x,y,z)$ with the help of relations

$$F_1 = \varphi_3^- - A_2\varphi_4^-, \qquad F_2 = \varphi_3^- - \frac{1}{A_2}\varphi_4^- \qquad (1.24)$$

From (1.19) and (1.23) we obtain for the functions a problem in potential theory

$$F_1(x, y, 0) = C, \qquad \left[\frac{\partial F_2(x, y, z)}{\partial z}\right]_{z=0} = C_2 \qquad (\rho < a) \qquad (1.25)$$

$$F_1(x,y,0) - AF_2(x,y,0) = 0, \ \left[\frac{\partial F_1(x, y, z)}{\partial z} - B\frac{\partial F_2(x, y, z)}{\partial z}\right]_{z=0} = 0 \qquad (\rho > a)$$

$$\left(A = \frac{A_2 - A_0}{1 - A_0A_2}A_2, \quad B = \frac{1 - A_0A_2}{A_2 - A_0}A_2\right)$$

2. Reduction of axially symmetric problem in potential theory in three dimensions to the subsidiary potential theory problem in the plane. In the half-space $z < 0$ two harmonic functions $F_1(x,y,z)$ and $F_2(x,y,z)$ are given subject to boundary conditions (1.25).

Functions $F_1(x,y,z)$ and $F_2(x,y,z)$, by virtue of the independence upon angle φ, are designated, correspondingly, $F_1(\rho,z)$ and $F_2(\rho,z)$ and we represent them in the form

$$F_k(\rho, z) = \int_0^\infty f_k(\alpha) J_0(\rho\alpha) e^{\alpha z} d\alpha \qquad (k = 1, 2) \qquad (2.1)$$

Differentiating (2.1) with respect to z, we get

$$\frac{\partial F_k(\rho, z)}{\partial z} = \int_0^\infty f_k(\alpha)\, \alpha J_0(\rho\alpha) e^{\alpha z} d\alpha \qquad (k = 1, 2) \qquad (2.2)$$

where $J_0(\alpha r)$ is the Bessel function of zero order. We make use of the representation of a Bessel function in the form of a contour integrals [2]

$$J_0(\rho\alpha) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{2^{-s}\Gamma(1/2 - 1/2 s)}{\Gamma(1/2 + 1/2 s)}\rho^{s-1}\alpha^{s-1}ds$$

$$\alpha J_0(\rho\alpha) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{2^{1-s}\Gamma(1 - 1/2 s)}{\Gamma(1/2 s)}\rho^{s-2}\alpha^{s-1}ds \tag{2.3}$$

Substituting these expressions in Equations (2.1) and (2.2), changing the order of integration in the two integrals obtained and designating

$$\int_0^\infty f_k(\alpha)\,\alpha^{s-1}e^{\alpha z}d\alpha = \Phi_k(s, z) \qquad (k = 1, 2) \tag{2.4}$$

we get

$$F_k(\rho, z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Phi_k(s, z)\frac{2^{-s}\Gamma(1/2 - 1/2 s)}{\Gamma(1/2 + 1/2 s)}\rho^{s-1}ds$$

$$\frac{\partial F_k(\rho, z)}{\partial z} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Phi_k(s, z)\frac{2^{1-s}\Gamma(1 - 1/2 s)}{\Gamma(1/2 s)}\rho^{s-2}ds \qquad (k = 1, 2) \tag{2.5}$$

We introduce functions $U_1(x, z)$ and $U_2(x, z)$, harmonic in the half-space $z < 0$, antisymmetric with respect to x, with the aid of the relation

$$U_k(x, z) = \int_0^\infty \frac{1}{\alpha} f_k(\alpha)\sin(\alpha x)\,e^{\alpha z}d\alpha \qquad (k = 1, 2) \tag{2.6}$$

Differentiating (2.6) by z, we obtain

$$\frac{\partial U_k(x, z)}{\partial z} = \int_0^\infty f_k(\alpha)\sin(\alpha x)\,e^{\alpha z}d\alpha \tag{2.7}$$

We substitute in (2.6) and (2.7), instead of functions $\alpha^{-1}\sin(\alpha x)$, $\sin(\alpha x)$ their representations

$$\frac{1}{\alpha}\sin(\alpha x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\sqrt{\pi}\,\frac{2^{1-s}\Gamma(1/2 - 1/2 s)}{\Gamma(1 + 1/2 s)}x^s\alpha^{s-1}ds$$

$$\sin(\alpha x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\sqrt{\pi}\,\frac{2^{2-s}\Gamma(1 - 1/2 s)}{\Gamma(1/2 + 1/2 s)}x^{s-1}\alpha^{s-1}ds \tag{2.8}$$

Changing the order of integration and taking advantage of (2.4) we get

$$U_k(x, z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\sqrt{\pi}\,\Phi_k(s, z)\frac{2^{1-s}\Gamma(1/2 - 1/2 s)}{\Gamma(1 + 1/2 s)}x^s ds$$

$$\frac{\partial U_k(x, z)}{\partial z} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\sqrt{\pi}\,\Phi_k(s, z)\frac{2^{2-s}\Gamma(1 - 1/2 s)}{\Gamma(1/2 + 1/2 s)}x^{s-1}ds \qquad (k = 1, 2) \tag{2.9}$$

From the well known formula

$$\int_0^1 x^{\alpha-1}(1 - x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

one easily obtains [3]

$$\int_0^x \rho^{2\alpha-1}(x^2-\rho^2)^{\beta-1}d\rho = \frac{1}{4}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}x^{2\alpha+2\beta-2}$$
$$\int_x^\infty \rho^{-2\alpha-2\beta+1}(\rho^2-x^2)^{\beta-1}\,d\rho = \frac{1}{2}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}x^{-2\alpha} \qquad (2.10)$$

Taking advantage of Equations (2.10), form (2.5) and (2.9) we find

$$\frac{\partial}{\partial x}\int_0^x \frac{F_k(\rho, z)}{\sqrt{x^2-\rho^2}}\rho d\rho = \frac{1}{2}\frac{\partial U_k(x, z)}{\partial x}, \qquad -\frac{\partial}{\partial x}\int_x^\infty \frac{F_k(\rho, z)}{\sqrt{\rho^2-x^2}}\rho d\rho = \frac{1}{4}\frac{\partial U_k(x, z)}{\partial z} \qquad (2.11)$$

$$\frac{1}{2\pi}\int_0^\rho \frac{\partial U_k(x, z)}{\partial x}\frac{dx}{\sqrt{\rho^2-x^2}} = F_k(\rho, z), \quad \frac{1}{2\pi\rho}\frac{\partial}{\partial\rho}\int_0^\rho \frac{\partial U_k(x, z)}{\partial z}\frac{xdx}{\sqrt{\rho^2-x^2}} = \frac{\partial F_k(\rho, z)}{\partial z} \qquad (2.12)$$

$$-\frac{1}{2\pi\rho}\frac{\partial}{\partial\rho}\int_\rho^\infty \frac{\partial U_k(x, z)}{\partial x}\frac{xdx}{\sqrt{x^2-\rho^2}} = \frac{\partial F_k(\rho, z)}{\partial z}, \quad \frac{1}{2\pi}\int_\rho^\infty \frac{\partial U_k(x, z)}{\partial z}\frac{dx}{\sqrt{x^2-\rho^2}} = F_k(\rho, z)$$
$$(k = 1, 2) \qquad (2.13)$$

On the basis of Equations (2.12), which are valid also for $z = 0$, boundary conditions (1.23) are reduced to the form

$$\frac{\partial U_1(x, 0)}{\partial x} = g_1(x), \qquad \left[\frac{\partial U_2(x, z)}{\partial z}\right]_{z=0} = g_2(x) \qquad (|x| < a)$$
$$\frac{\partial U_1(x, 0)}{\partial x} - B\frac{\partial U_2(x, 0)}{\partial x} = 0, \qquad \left[\frac{\partial U_1(x, z)}{\partial z} - A\frac{\partial U_2(x, z)}{\partial z}\right]_{z=0} = 0 \qquad (|x| > a) \qquad (2.14)$$

Here

$$g_1(x) = 4\frac{d}{dx}\int_0^x \frac{C\rho d\rho}{\sqrt{x^2-\rho^2}}, \qquad g_2(x) = 4\int_0^x \frac{C_2\rho d\rho}{\sqrt{x^2-\rho^2}} \qquad (0 < x < a)$$

Thus, the axially symmetric problem of potential theory, formulated in the beginning of this Section, is reduced to the following plane problem in potential theory: find the value of functions $U_1(x,z)$ and $U_2(x,z)$ which are harmonic in the half-plane $z < 0$, if boundary conditions (2.14) are given.

3. Solution of the plane problem in potential theory. In the sequel, following Muskhelishvili [4], the half-plane $z < 0$ we designate by S^-, the upper half-plane $z > 0$ we designate by S^+. The interval $-a < x < a$ on axis Ox we designate by L', the remaining portion of the axis is L''. The functions $U_1(x,z)$ and $U_2(x,z)$, harmonic in the half-plane $z < 0$, will be regarded as the real parts of analytic functions $\Phi_1(\zeta)$ and $\Phi_2(\zeta)$ ($\zeta = x + iz$)

$$U_k(x, z) = {}^1/_2\Phi_k(\zeta) + {}^1/_2\overline{\Phi_k(\zeta)} \qquad (k = 1, 2) \qquad (3.1)$$

From (3.1) we obtain

$$\frac{\partial U_k}{\partial x} = \frac{1}{2}\Phi_k'(\zeta) + \frac{1}{2}\overline{\Phi_k'(\zeta)}, \quad \frac{\partial U_k}{\partial z} = \frac{i}{2}\Phi_k'(\zeta) - \frac{i}{2}\overline{\Phi_k'(\zeta)} \qquad (k = 1, 2) \quad (3.2)$$

Setting z to zero in Equation (3.2) and substituting the value of functions $\partial U_k/\partial x$ and $\partial U_k/\partial z$ in the conditions (2.14), we get boundary conditions for the determination of functions $\Phi_1(\zeta)$ and $\Phi_2(\zeta)$ in the form

$$\left.\begin{aligned} \Phi_1'^- + \overline{\Phi_1'^+} &= 2g_1(x) \\ \Phi_2'^- - \overline{\Phi_2'^+} &= -2ig_1(x) \end{aligned}\right\} \text{ on } L'$$

$$\left.\begin{aligned} \Phi_1'^- + \overline{\Phi_1'^+} - B\Phi_2'^- - B\overline{\Phi_2}^+ &= 0 \\ \Phi_1'^- - \overline{\Phi_1'^+} - A\Phi_2'^- + A\overline{\Phi_2'^+} &= 0 \end{aligned}\right\} \text{ on } L'' \tag{3.3}$$

Let us introduce functions $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$, analytic over the entire plane ζ, with the exception of a slit, coinciding with L', by means of relations

$$\begin{aligned} &\Phi_1'(\zeta) - B\Phi_2'(\zeta) = \Omega_1(\zeta), \qquad \Phi_1'(\zeta) - A\Phi_2'(\zeta) = \Omega_2(\zeta) \text{ в } S^- \\ &\overline{\Phi}_1'(\zeta) - B\overline{\Phi}_2'(\zeta) = -\Omega_1(\zeta), \quad \overline{\Phi}_1'(\zeta) - A\overline{\Phi}_2'(\zeta) = \Omega_2(\zeta) \text{ в } S^+ \end{aligned} \tag{3.4}$$

Boundary conditions for $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$ on L' are

$$\begin{aligned} &\frac{A}{A-B}\Omega_1^- - \frac{B}{A-B}\Omega_2^- - \frac{A}{A-B}\Omega_1^+ - \frac{B}{A-B}\Omega_2^+ = 2g_1(x) \\ &\frac{1}{A-B}\Omega_1^- - \frac{1}{A-B}\Omega_2^- + \frac{1}{A-B}\Omega_1^+ + \frac{1}{A-B}\Omega_2^+ = -2ig_2(x) \end{aligned} \quad \text{on } L' \tag{3.5}$$

Conditions on L'' are satisfied by the selection of functions $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$. Equation (3.5) is reduced to the problem of linear relationship

$$\left[\Omega_1(x) - \frac{B+\sqrt{AB}}{A+\sqrt{AB}}\Omega_2(x)\right]^- - \frac{A-\sqrt{AB}}{A+\sqrt{AB}}\left[\Omega_1(x) - \frac{B+\sqrt{AB}}{A+\sqrt{AB}}\Omega_2(x)\right]^+ = \frac{2(A-B)}{A+\sqrt{AB}}[g_1(x) - i\sqrt{AB}g_2(x)]$$

$$\left[\Omega_1(x) - \frac{B-\sqrt{AB}}{A-\sqrt{AB}}\Omega_2(x)\right]^- - \frac{A+\sqrt{AB}}{A-\sqrt{AB}}\left[\Omega_1(x) - \frac{B-\sqrt{AB}}{A-\sqrt{AB}}\Omega_2(x)\right]^+ = \frac{2(A-B)}{A-\sqrt{AB}}[g_1(x) + i\sqrt{AB}g_2(x)] \tag{3.6}$$

Solving the problem of linear relationship we find

$$\Omega_1(\zeta) = 2(A-B)\left\{\left(2a\gamma C_2 i - \frac{C}{\sqrt{AB}}\right)\left[\left(\frac{\zeta-a}{\zeta+a}\right)^{\gamma} - \left(\frac{\zeta-a}{\zeta+a}\right)^{-\gamma}\right] + C_2 i\zeta\left[\left(\frac{\zeta-a}{\zeta+a}\right)^{\gamma} + \left(\frac{\zeta-a}{\zeta+a}\right)^{-\gamma}\right] - 2C_2 i\zeta\right\} \tag{3.7}$$

$$\Omega_2(\zeta) = \frac{2A(A-B)}{\sqrt{AB}}\left\{C_2 i\zeta\left[\left(\frac{\zeta-a}{\zeta+a}\right)^{-\gamma} - \left(\frac{\zeta-a}{\zeta+a}\right)^{\gamma}\right] - \left(2a\gamma C_2 i - \frac{C}{\sqrt{AB}}\right)\left[\left(\frac{\zeta-a}{\zeta+a}\right)^{-\gamma} + \left(\frac{\zeta-a}{\zeta+a}\right)^{\gamma}\right] - \frac{2C}{\sqrt{AB}}\right\} \tag{3.8}$$

$$\left(\gamma = \frac{1}{2\pi i}\ln\frac{A+\sqrt{AB}}{A-\sqrt{AB}}\right)$$

After functions $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$ are determined, boundary values for functions $\partial U_k/\partial x$ and $\partial U_k/\partial z$ are easily found.

The constant C is found from the following condition.

Functions $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$ being derivatives of regular functions at infinity, must vanish at least as z^{-1}. This gives

$$C = \sqrt{AB}a\gamma C_2 i \tag{3.9}$$

4. The determination of the critical stress, normal and shear stresses exterior to the crack on the plane of joining. The existence in the body of a circular crack of radius a reduces its potential energy by an amount

$$W = \frac{1}{2}\iint\limits_{\sigma} \rho\,(w^+ - w^-)\,d\sigma \tag{4.1}$$

where w^+, w^- are the displacements on the lower and upper boundary of the gap, the domain of integration σ is the circle of radius a.

In addition, the crack has surface energy U, equal to

$$U = 2\pi a^2 T_0 \tag{4.2}$$

where T_0 is the specific surface energy. According to Griffith [5], the condition necessary for the enlargement of the crack consists of the following

$$\frac{\partial}{\partial a}(W - U) = 0 \tag{4.3}$$

From relation (4.3) we get the criterion of failure.

Displacements w^+ and w^- on the basis of (1.4) are

$$w^+(x, y, 0) = \varphi_3^+(x, y, 0), \qquad w^-(x, y, 0) = \varphi_3^-(x, y, 0)$$

Integral (4.1) may be written in the form

$$W = \tfrac{1}{2}\, p\int_0^{2\pi} d\varphi \int_0^a (\varphi_3^+ - \varphi_3^-)\,\rho d\rho = \pi\rho \int_0^a (\varphi_3^+ - \varphi_3^-)\,\rho d\rho \tag{4.4}$$

Making use of (3.7), (3.8), (3.2), (2.11) and (1.21), we find (4.5)

$$\varphi_3^- = \frac{C_2}{A_2 - 1/A_2}\Big(\frac{A_2\sqrt{B(A-B)}}{\pi B}\,i\int_0^{\rho}\Big\{a\gamma\Big[\Big(\frac{a-x}{a+x}\Big)^{\gamma} + \Big(\frac{a-x}{a+x}\Big)^{-\gamma}\Big] + \Big[\Big(\frac{a-x}{a+x}\Big)^{\gamma} - \Big(\frac{a-x}{a+x}\Big)^{-\gamma}\Big]x\Big\}\frac{dx}{\sqrt{\rho^2 - x^2}} + \Big(\frac{A_2}{B} - \frac{1}{A_2}\Big)a\gamma\sqrt{AB}i\Big)$$

$$\varphi_3^+ = \frac{C_2}{A_2 - 1/A_2}\Big(\frac{\sqrt{B(A-B)}}{\pi B}(HA_2 - H)\,i\int_0^{\rho}\Big\{a\gamma\Big[\Big(\frac{a-x}{a+x}\Big)^{\gamma} + \Big(\frac{a-x}{a+x}\Big)^{-\gamma}\Big] + x\Big[\Big(\frac{a-x}{a+x}\Big)^{\gamma} - \Big(\frac{a-x}{a+x}\Big)^{-\gamma}\Big]\Big\}\frac{dx}{\sqrt{\rho^2 - x^2}} + \Big(\frac{A_2E}{B} - \frac{E}{A_2} - \frac{H}{B} + H\Big)\sqrt{AB}a\gamma i\Big)$$

Substituting the value of (4.5) in (4.4) and changing the order of integration we obtain

$$W = \frac{\pi\rho C_2 i}{A_2 - 1/A_2}\left\{\left[H - \frac{E-1}{A_2} - \frac{H - A_2'(E-1)}{B}\right]\sqrt{AB}\gamma\frac{a^3}{2} + \right.$$
$$+ [A_2(E-1) - H]\frac{\sqrt{B(A-B)}}{\pi B}\int_0^a \left(a\gamma\left[\left(\frac{a-x}{a+x}\right)^{\gamma} + \left(\frac{a-x}{a+x}\right)^{-\gamma}\right] + \right.$$
$$\left.\left. + x\left[\left(\frac{a-x}{a+x}\right)^{\gamma} - \left(\frac{a-x}{a+x}\right)^{-\gamma}\right]\right)\sqrt{a^2 - x^2}dx\right\} \tag{4.6}$$

The integrals entering in (4.6) are computed by means of the relationship [2]

$$\int_0^1 x^{\alpha-1}(1 - x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{4.7}$$

We have

$$\int_0^a \left[\left(\frac{a-x}{a+x}\right)^{\gamma} + \left(\frac{a-x}{a+x}\right)^{-\gamma}\right]\sqrt{a^2 - x^2}dx = 4a^2\,\frac{\Gamma(^3/_2+\gamma)\Gamma(^3/_2-\gamma)}{\Gamma(3)}$$
$$\int_0^a x\left[\left(\frac{a-x}{a+x}\right)^{\gamma} - \left(\frac{a-x}{a+x}\right)^{-\gamma}\right]\sqrt{a^2 - x^2}dx = \frac{4a^3}{\Gamma(4)}\left[\Gamma\left(\frac{5}{2}-\gamma\right)\Gamma\left(\frac{3}{2}+\gamma\right) - \right.$$
$$\left. -\Gamma\left(\frac{5}{2}+\gamma\right)\Gamma\left(\frac{3}{2}-\gamma\right)\right] \tag{4.8}$$

We calculate the value of gamma functions by means of the relations

$$\Gamma(1+z) = z\Gamma(z), \qquad \Gamma(z)\Gamma(1-z) = \pi\csc\pi z \tag{4.9}$$

and we substitute the value of (4.8) into (4.6). Then we get

$$W = \frac{\pi p C_2}{A_2 - A_2^{-1}}\left\{\frac{1}{2}\left[H - \frac{E-1}{A_2} - \frac{H - A_2(E-1)}{B}\right]\sqrt{AB} + \right.$$
$$\left. + \frac{3}{2}[H - A_2(E-1)]\frac{\sqrt{B(A-B)}}{B}\left(\gamma^2 - \frac{1}{4}\right)\frac{1}{\cos\pi\gamma}\right\}a^3\gamma \tag{4.10}$$

Taking into account that

$$\cos\pi\gamma = \frac{A}{\sqrt{A^2 - A_2^2}}, \quad \lambda_k = \frac{E_k\nu_k}{(1+\nu_k)(1-2\nu_k)}, \quad \mu_k = \frac{E_k}{2(1+\nu_k)} \qquad (k = 1,2)$$

and substituting the value of the constants from (1.9), (1.22), (1.23) and (1.25), we get Equations

$$W = -\frac{2\pi p^2}{3}\frac{\vartheta_2 E_1^2 + \vartheta_1 E_2^2 + 2E_1\vartheta_{12}E_2(1+\nu_2)(1+\nu_2)}{E_1E_2[E_1(1+\nu_2)(1-2\nu_2) - E_2(1+\nu_1)(1-2\nu_1)]}(\Theta^2+1)\Theta a^3 \tag{4.11}$$

$$\Theta = \frac{1}{2\pi}\ln\frac{E_1(1+\nu_2)(3-4\nu_2) + E_2(1+\nu_1)}{E_1(1+\nu_2) + E_2(1+\nu_1)(3-4\nu_1)}$$
$$\vartheta_{12} = (1-2\nu_1)(1-2\nu_2) + 4(1-\nu_1)(1-\nu_2) \tag{4.12}$$
$$\vartheta_k = (1+\nu_k)^2(3-4\nu_k) \quad (k = 1, 2)$$

Here E_1 and E_2 are Young's moduli for the upper and lower half-spaces and ν_1 and ν_2 are Poisson's coefficients.

Substituting W from (4.11) and U from (4.2) in the relation (4.3), we find the critical stresses depending on the radius of the crack

$$p_0 = \left(\frac{2T_0E_1E_2\,[E_1(1+\nu_2)(1-2\nu_2) - E_2(1+\nu_1)(1-2\nu_1)]}{[\vartheta_2E_1^2 + \vartheta_1E_2^2 + 2\vartheta_{12}E_1E_2(1+\nu_1)(1+\nu_2)]\,(\Theta^2+1)\,\Theta a}\right)^{1/2} \qquad (4.13)$$

Hence, in the case of an isotropic body $(E_1 = E_2 = E,\ \nu_1 = \nu_2 = \nu)$ the result of Sack [6] follows

$$p_0 = \sqrt{\frac{\pi T_0 E}{2(1-\nu^2)\,a}} \qquad (4.14)$$

Sack has extended the Griffith theory of failure to the three-dimensional case. In the special case where one of the half-spaces is absolutely rigid $(E_1 = \infty)$, the collapse occurs when

$$p_0 = \left(\frac{2\pi T_0 E}{a(1+\nu_2)(3-4\nu_2)\,[{}^1/_4\pi^{-2}\ln^2(3-4\nu_2)+1]\ln(3-4\nu_2)}\right)^{1/2} \qquad (4.15)$$

Using (1.3), (1.23), (2.13), (3.7), (3.9) and carrying out certain transformations, we obtain the formulas for the determination of normal and shear stresses exterior to the crack on the plane of joining (4.16)

$$\sigma_z = \frac{4p}{\pi}\,\frac{E_1(1-\nu_2^2) + E_2(1-\nu_1^2)}{E_1(1+\nu_2)(1-2\nu_2) - E_2(1+\nu_1)(1-2\nu_1)}\int_1^0 \left[\left(\frac{2}{t} - \frac{2a^2\Theta^2 t}{\rho^2 - a^2t^2}\right)\times\right.$$

$$\left.\times\sin\left(\Theta\ln\frac{\rho - at}{\rho + at}\right) + \left(\frac{2\rho}{\rho^2 - a^2t^2} + \frac{1}{\rho}\right)a\Theta\cos\left(\Theta\ln\frac{\rho - at}{\rho + at}\right) + \frac{a\Theta}{\rho}\right]\frac{dt}{t^2\sqrt{1-t^2}}$$

$$\tau_{\rho z} = \frac{4p}{\pi}\,\frac{E_2(1-\nu_2^2) + E_2(1-\nu_1^2)}{E_1(1+\nu_2)(1-2\nu_2) - E_2(1+\nu_1)(1-2\nu_1)}\int_1^0 \left[\left(\frac{2a^2\Theta^2 t}{\rho^2 - a^2t^2} - \frac{1}{t}\right)\times\right.$$

$$\left.\times\cos\left(\Theta\ln\frac{\rho - at}{\rho + at}\right) + \frac{2\rho a\Theta}{\rho^2 - a^2t^2}\sin\left(\Theta\ln\frac{\rho - at}{\rho + at}\right) + \frac{1}{t}\right]\frac{dt}{t\sqrt{1-t^2}} \qquad (4.17)$$

Integrals in (4.16) and (4.17) are obtained by numerical evaluation.

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